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Fuzzy weighted averages and fuzzy convex sums: Author's response

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Abstract

We elaborate further on the reasons for the lack of equality between interval weighted averages and interval-based convex sums and highlight its consequences on the cogency of multicriteria decision evaluation procedures based on the use of fuzzy interval weights.

Keywords: Interval analysis; Weighted average; Convex sums

In his note, Pavlačka [3] corrects a technical mistake appearing in my position paper [4]. I am grateful for this correction for two reasons:

- The geometrical analysis of constraining ill-known normalized vectors vs. normalizing ill-known weights is very instructive on the difference between the two methods.
- Moreover, this result seems to raise an interesting issue in the use of interval and fuzzy extensions of weighted averages in decision analysis.

From a geometrical point of view, the problem is to study the relative positions of the sets of probability vectors

$$\mathbf{W} = \left\{ \left(\frac{w_1}{\sum_{i=1}^n w_i}, \ldots, \frac{w_n}{\sum_{i=1}^n w_i} \right) : w_i \in [a_i, b_i], i = 1, \ldots, n \right\}$$

and

$$\mathbf{W}^N = \left\{ (p_1, \ldots, p_n) : p_i \in \left[\frac{a_i}{a_i + \sum_{j \neq i} b_j}, \frac{b_i}{b_i + \sum_{j \neq i} a_j} \right], i = 1, \ldots, n, \sum_{i=1}^n p_i = 1 \right\}.$$

For $n = 2$ these sets are equal. Consider the case $n > 2$ and for simplicity we assume $a_i > 0, \forall i$. It can be observed that the polyhedra $\mathbf{W}$ and $\mathbf{W}^N$ both lie in the hyperplane $\mathbf{H} = \{\vec{p} = (p_1, \ldots, p_n) : \sum_{i=1}^n p_i = 1\}$ of normalized vectors, but no facet of one is parallel to a facet of the other. Let $\mathbf{P} = \{\vec{p} \in \mathbf{H} : p_i \geq 0, \ i = 1, \ldots, n\}$ be the positive part of $\mathbf{H}$, that is, the set of probability assignments in $\mathbf{H}$.

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It is clear that $\mathbf{W}^N$ is the intersection between $\mathbf{P}$ and the Cartesian product defined by $\times_{i=1,\ldots,n}[a_i/(a_i + \sum_{j\neq i} b_j),$ $b_i/(b_i + \sum_{j\neq i} a_j)]$. Each pair of parallel facets of $\mathbf{W}^N$ of the form $p_i = p$, where $p = a_i/(a_i + \sum_{j\neq i} b_j)$ and $p = b_i/(b_i + \sum_{j\neq i} a_j)$ is also parallel to the facet of $\mathbf{P}$ such that $p_i = 0$, and this for $i = 1, \ldots, n$. In contrast, $\mathbf{W}$ is obtained by homothetic projection, with center equal to the origin $(0, \ldots, 0)$, of the Cartesian product $\times_{i=1,\ldots,n}[a_i, b_i]$ on $\mathbf{P}$. The vertices of this polyhedron lie inside the facets of $\mathbf{W}^N$ as shown in [3], so that both $\mathbf{W}^N$ and $\mathbf{W}$ have the same marginal projections of the form $[a_i/(a_i + \sum_{j\neq i} b_j), b_i/(b_i + \sum_{j\neq i} a_j)]$. Moreover, each edge of $\mathbf{W}$ is the homothetic projection of one edge of the hyper-rectangle $\times_{i=1,\ldots,n}[a_i, b_i]$ and lies in the same hyperplane as one axis of the referential, hence on a line containing the vertex of $\mathbf{P}$ on that axis. The set $\mathbf{W}$ is defined via constraints of the form $p_i/p_j = w_i/w_j$, while $\mathbf{W}^N$ is generated from bounds on individual p_i. This type of representation is common in the area of imprecise probabilities [5], where credal sets of the form $\mathbf{W}^N$ are obtained by constraints on the probabilities of elementary events [2] while those of the form $\mathbf{W}$ are obtained by constraining conditional probabilities.

For instance consider the case where $n = 3$ pictured by Fig. 1 in Pavlačka [3], where $\mathbf{P}$ is an equilateral triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The set $\mathbf{W}$ of renormalized weights is a hexagon whose sides are obtained by lines drawn from the vertices of the triangle (two per vertex). In contrast, $\mathbf{W}^N$ is the smallest hexagon, with opposite edges parallel to the edges of the triangle $\mathbf{P}$, containing $\mathbf{W}$. In the fuzzy case, we get fuzzy subsets of $\mathbf{P}$ consisting of nested hexagons.

It should be clear from [3] and the above additional comments that the vertices of $\mathbf{W}$ are generally not the same as the vertices of $\mathbf{W}^N$. As a consequence, the intervals

$$\mathbf{X} = \left\{ \frac{\sum_{i=1}^{n} x_i \cdot w_i}{\sum_{i=1}^{n} w_i} : w_i \in [a_i, b_i], i = 1, \ldots, n \right\}$$

and

$$\mathbf{X}^N = \left\{ \sum_{i=1}^{n} x_i \cdot p_i : p_i \in \left[\frac{a_i}{a_i + \sum_{j\neq i} b_j}, \frac{b_i}{b_i + \sum_{j\neq i} a_j} \right], i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1 \right\}$$

have little chance to be equal because their bounds are attained on vertices of $\mathbf{W}$ and $\mathbf{W}^N$ respectively. And as $\mathbf{W} \subset \mathbf{W}^N$ it follows that, in general, $\mathbf{X} \subset \mathbf{X}^N$.

A consequence of this finding is that in multicriteria decision-making based on fuzzy weighted averages, constraining ill-known normalized vectors and normalizing ill-known weights will give different results. Note that in the precise case, one may indifferently compute a convex sum of ratings according to the various criteria either using normalized weight vectors or using a weighted sum with suitable non-normalized weight vectors. As the results without normalization will be proportional to the results using a normalized vector, this choice of method is immaterial for determining the ranking of available options. However in the interval-valued case (let alone in the fuzzy case) there is no guarantee that the rankings of options will be the same with the two approaches, respectively using intervals $\mathbf{X}$ and $\mathbf{X}^N$ for comparing the options, as they rely on imprecision polyhedra having different vertices. All we know is that $\mathbf{X}$ will be strictly included in $\mathbf{X}^N$. Even worse, there is generally no way of having the polyhedron of normalized vectors induced by unnormalized interval weights equal to a polyhedron induced by suitable ranges of imprecision bearing on components of a normalized weight vector, due to the general incompatibility of their respective configurations.

This dilemma is very problematic when applying fuzzy weighted averages to multicriteria decision analysis. Namely, how to model the imprecision on the weights bearing on each criterion? Usually a weighted average is defined as a convex sum of ratings. Should we elicit imprecision on normalized weight vectors and compute a fuzzy convex sum? or elicit imprecision on unnormalized weights and compute the image of the fuzzy weights via a rational fraction as done by most authors since Baas and Kwakernaak [1] proposed it? The useful correction of Pavlačka [3] should lead to a reappraisal of the cogency of the fuzzy weighted average approach to decision analysis.

The natural way out of this dilemma may come from the following considerations.[1] Note that the two intervals $\mathbf{X}_1$ and $\mathbf{X}_2$ (likewise $\mathbf{X}_1^N$ and $\mathbf{X}_2^N$) evaluating two options 1 and 2 do not constrain independent quantities: Each assignment of weights w_i in $[a_i, b_i]$ determines single values, one in $\mathbf{X}_1$ and one in $\mathbf{X}_2$, that are linked to each other. So the final ranking of options based on (fuzzy) interval global ratings cannot be achieved by applying any (fuzzy) interval

[1] Based on a suggestion by Eyke Huellermeier, gratefully acknowledged.

ranking method to one of the sets of ill-known ratings $\mathbf{X}_k$ or $\mathbf{X}_k^N$. The interval ratings obtained for each option are not independent. A natural ranking criterion is then the following: If one option is better than another one for each assignment of weights in their ranges, then the former option is better than the latter. Note that this ranking criterion yields results that the renormalization procedure will leave unchanged: with this criterion, comparing $\mathbf{X}_1$ and $\mathbf{X}_2$, or $\mathbf{X}_1^N$ and $\mathbf{X}_2^N$, would give the same results. As pointed out by a reviewer of this note, this fact pleads in favor of extending, to intervals or fuzzy intervals, the whole decision procedure at once, including the aggregation and the ranking steps, so as to take the interactivity between the fuzzy global evaluation intervals into account.

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